Previous Final exam on Moodle (Answers)

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MATHEMATICS 218
Final Examination

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ID# ---------

Circle your section number :

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PART I. Answer each of the following problems in the space provided for each problem (Problem 1 to Problem 5).

1. Let $A=\begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

(a) Find the eigenvalues and a basis for each eigenspace of A. [10 points]

$|A-\lambda I|=0 \Rightarrow \begin{vmatrix} 5-\lambda & 1 & 0 \\ 1 & 5-\lambda & 0 \\ 0 & 0 & 6-\lambda \end{vmatrix}=0 \Rightarrow (6-\lambda)\begin{vmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix} = (6-\lambda)(25-10\lambda+\lambda^2-1) = (6-\lambda)(6-\lambda)(4-\lambda)$

So $\lambda_1=\lambda_2=6$, $\lambda_3=4$

* Basis for Eigenspace $(V_{\lambda_1=\lambda_2=6}) = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

* Basis for Eigenspace $(V_{\lambda_3=4}) = \left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$

So $\mathbb{R}^3$ has 3 linearly indep. eigenvectors
So diagonalizable

1(b) Show that A is diagonalizable. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP=D$. (Do not verify) [6 points]

Let $P=\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $D=\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$

$\lambda_1=\lambda_2=6$, $\lambda_3=4$

$P^{-1}AP = D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ diagonal matrix

2. Let $T : P_2 \to P_2$ be the linear transformation defined by $T(p(x)) = xp'(x)$.

(a) Find the matrix $A=[T]_\beta$ of T relative to the standard ordered basis $\beta = \{1, x, x^2\}$ of P_2.

[6 points]

$T(1) = x(1)' = 0 = 0.1 + 0.x + 0.x^2$

$T(x) = x(x)' = x = 0.1 + 1.x + 0.x^2$

$T(x^2) = x(x^2)' = 2x^2 = 0.1 + 0.x + 2.x^2$

So $A = [T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(b) Find the matrix $B=[T]_{\beta'}$ of T relative to the ordered basis $\beta'=\{1+x^2, 2x, 1\}$ of P_2.

[6 points]

$T(1+x^2) = x(2x) = 2x^2 = 2.(1+x^2) + 0.2x + (-2).1$

$T(2x) = x(2x)' = 2x = 0.(1+x^2) + 1.2x + 0.1$

$T(1) = x.(0) = 0 = 0.(1+x^2) + 0.2x + 0.1$

So $B = [T]_{\beta'} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$

(c) Find the transition matrix $P=[I]_{\beta'}^{\beta}$ from β' to β such that $B=P^{-1}AP$ (Do not verify).

[6 points]

$\beta' = \{1+x^2, 2x, 1\}$ old

$\beta = \{1, x, x^2\}$ new

old.

$I(1+x^2) = 1+x^2 = 1.1 + 0.x + 1.x^2$ $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$I(2x) = 2x = 0.1 + 2.x + 0.x^2$ $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$I(1) = 1 = 1.1 + 0.x + 0.x^2$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$P = [I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

3. Prove that an orthogonal set of nonzero vectors in an inner product space V is linearly independent.

[10 points]

SEE NOTES

4. Let T: $R^3 \to R^3$ be the linear transformation defined by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b+c \\ 0 \\ 0 \end{pmatrix}.$$

Find a basis for the null space N(T) and use the Gram-Schmidt process to construct an orthonormal basis for N(T), with the usual dot product.

[12 points]

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in N(T) \Rightarrow T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b+c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So $a+b+c=0 \Rightarrow c=-a-b$

So $N(T) = \left\{ \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix} \right\}$ 2 parameters a, b

Basis for $N(T) = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}} \right\}$

Want orthogonal basis

Let $w_1 = v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. By Gram-schmidt

$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix}$

So orthogonal basis of $N(T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix} \right\}$

5. Let T: $R^3 \to R^4$ and S: $R^4 \to R^2$ be two linear transformations such that the composition SoT =0. Show that if S is onto then T cannot be one-to-one.

$\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^4 \xrightarrow{S} \mathbb{R}^2$ $\quad S \circ T = 0$

[7 points]

So $S \circ T(v) = 0 \Rightarrow S(T(v)) = 0 \Rightarrow R(T) \subseteq N(S)$

So $\dim R(T) \le \dim N(S)$

S onto $\Rightarrow$ Range $(S) = \mathbb{R}^2 \Rightarrow \dim R(S) = 2$

$\dim N(S) + \dim R(S) = \dim \mathbb{R}^4 = 4 \Rightarrow \dim N(S) = 4-2 = 2$

Suppose T is one to one, Want a contradiction

$\dim N(T) + \dim(R(T)) = 3$ but $N(T) = \{0\}$

so $\dim(R(T)) = 3$ but $\underset{\dim 3}{R(T)} \subseteq \underset{\dim 2}{N(S)} \Rightarrow 3 \le 2$ Contradiction

So T cannot be 1-1.

PART II. **Circle the correct answer for each of the following problems (Problem 6 to Problem 12). IN THE TABLE IN THE FRONT PAGE [3 points for each correct answer]**

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x+2y \\ -x \\ 0 \end{pmatrix}.$$

$R(T) = \left\{ \begin{pmatrix} 3x+2y \\ -x \\ 0 \end{pmatrix} \right\}$ x, y parameters

$\dim R(T) = 2$

Then dim (Range T) is :

a. 3
b. 2
c. 1
d. none of the above.

[3 points]

7. Let S be the subspace defined by
S={M is a 3x3 skew-symmetric matrix with the sum of the entries of each row is zero}.
Then dim S=

$M^T = -M \Rightarrow M = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$

$a + b = 0 \Rightarrow b = -a$
$-a + c = 0 \Rightarrow c = a$
$-b - c = 0 \Rightarrow c = -b = a$

a. 1
b. 2
c. 3
d. none of the above.

[3 points]

So $M = \begin{pmatrix} 0 & a & -a \\ -a & 0 & a \\ a & -a & 0 \end{pmatrix}$

$\dim S = 1$

8. If T: $\mathbb{R}^2 \to \mathbb{R}$ is a linear transformation such that $T(v) \neq 0$ for some v in $\mathbb{R}^2$, then

$\dim(N(T)) + \dim(\text{Range } T) = 2$
$\dim(N(T)) \neq 0$

a. T is one-to-one.
b. T is onto.
c. Dim (Nullspace T)=0.
d. none of the above.

[3 points]

$\text{Range}(T) \subset \mathbb{R}^1$
$\Rightarrow \dim(\text{Range } T) \leq 1$

$\text{Range } T \neq \{0\}$ since $T(v) \neq 0$

So $\text{Range } T = \mathbb{R}$
T onto

PART III. Answer TRUE or FALSE only **IN THE TABLE IN THE FRONT PAGE** (**2 points for each correct answer**)

a. T ----- If A is a 3×3 matrix such that $A^2=0$, then rank A $\neq 3$. if rank A = 3 ⟹ A invertible so rank $A^2 \neq 0$

b. T --- Let V be a finite dimensional vector space and let W be a subspace of V. If dim W= dim V, then W=V. W subspace of V having same dimension as V

c. T--- If A is a 2×5 matrix, then dim (Column space of A) ≤ 2 dim (N(A)) + dim (Col space) = 2 So dim (Col A) ≤ 2

d. T---- Let V be a finite dimensional inner product space and let W be a subspace of V. Then the orthogonal complement of $W^{\perp}$ is equal to W. orthog. complement of $W^{\perp} = (W^{\perp})^{\perp} = W$

e. T------ Let V be a finite-dimensional vector space and let T: V→V be a linear transformation. If T is one-to-one, then T is onto. dim N(T) + dim R(T) = dim V ⟹ R(T) = ...

f. F---- The matrices A= $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ and B= $\begin{pmatrix} -2 & 1 \\ 4 & 3 \end{pmatrix}$ are similar. $|A| = 10$, $|B| = -10$

g. F---- Let T: $\mathbf{R}^2 \rightarrow \mathbf{R}^3$ be a linear transformation such that

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ and } T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

$$\begin{pmatrix} 5 \\ -1 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 5 \\ -1 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + -1\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Then $T\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -2 \end{pmatrix}$.

$$= \begin{pmatrix} 4 \\ 7 \\ 2 \end{pmatrix}$$

h. F---- The set of all 2×2 noninvertible matrices is a subspace of $M_{2\times2}$.

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{non invertible}} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{noninvertible}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{is invertible}}$$

[16 points].

9. Let $A=\begin{pmatrix}2 & 2\\ 1 & 2\\ 1 & 1\end{pmatrix}$ and $b=\begin{pmatrix}-2\\ 0\\ 1\end{pmatrix}$. Then the least squares solution to $Ax=b$ is:

$A^TAx = A^Tb$

$\begin{pmatrix}2 & 1 & 1\\ 2 & 2 & 1\end{pmatrix}\begin{pmatrix}2 & 2\\ 1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}2 & 1 & 1\\ 2 & 2 & 1\end{pmatrix}\begin{pmatrix}-2\\ 0\\ 1\end{pmatrix}$

$\begin{pmatrix}6 & 7\\ 7 & 9\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-3\\ -3\end{pmatrix}$

a. $\hat{x}=\begin{pmatrix}3/5\\ 6/5\end{pmatrix}$

b. $\hat{x}=\begin{pmatrix}-6/5\\ 3/5\end{pmatrix}$

c. $\hat{x}=\begin{pmatrix}-3/5\\ 6/5\end{pmatrix}$

d. None of the above

$\left.\begin{array}{l}6x+7y=-3\\ 7x+9y=-3\end{array}\right\} \Rightarrow \begin{array}{l}x=-6/5\\ y=3/5\end{array}$

[3 points]

10. Let U be the subspace of R^3 defined by

$U = \left\{\begin{pmatrix}x\\ y\\ z\end{pmatrix} \in R^3 \mid x-2y+3z=0\right\}$. Then $\dim U^{\perp} =$

$x = 2y-3z \Rightarrow U = \left\{\begin{pmatrix}2y-3z\\ y\\ z\end{pmatrix}\right\} \Rightarrow \dim U = 2$

$\Rightarrow \dim U^{\perp} = 3-2 = 1$

a. 3

b. 1

c. 2

d. none of the above

[3 points]

11. Let $A=\begin{pmatrix}1 & 1 & 2\\ 0 & 1 & 8\\ 0 & 1 & 3\end{pmatrix}$. Then:

$\begin{vmatrix}1-\lambda & 1 & 2\\ 0 & 1-\lambda & 8\\ 0 & 1 & 3-\lambda\end{vmatrix} = 0$

$(1-\lambda)(3-4\lambda+\lambda^2-8)=0$

$(1-\lambda).(\lambda-5).(\lambda+1)=0$

a. A is diagonalizable.

b. A is not invertible.

c. The eigenspace corresponding to the eigenvalue $\lambda=1$ has dimension 2.

d. None of the above.

$\lambda_1=1,\ \lambda_2=5,\ \lambda_3=-1$

3 distinct eigen values

$\Rightarrow$ diagonalizable

[3 points]

12. Let T: V→V be a linear transformation with dim V= n such that T is onto. Which one of the following statements is FALSE:

a. T is an isomorphism True $\dim N(T) + \dim R(T) = n = \dim R(T)$ (onto) $\Rightarrow N(T)=0 \Rightarrow$ 1-1 one to one

b. If { $v_1, v_2, \ldots, v_n$} is linearly independent in V, then True
{$T(v_1)$, $T(v_2)$, …, $T(v_n)$} is linearly independent in V. $c_1T(v_1)+\cdots+c_nT(v_n)=0 \Rightarrow T(c_1v_1+\cdots+c_nv_n)$

$\Rightarrow c_1v_1+\cdots+c_nv_n \in N(T) = 0$

$\Rightarrow c_1=\cdots=c_n=0$

c. dim(RangeT)=n True given onto

d. none of the above.

[3 points]